

# Enlarged spectral problems and nonintegrability

SERGEI SAKOVICH

Institute of Physics, National Academy of Sciences,  
220072 Minsk, Belarus. E-mail: saks@tut.by

## Abstract

The method of obtaining new integrable coupled equations through enlarging spectral problems of known integrable equations, which was recently proposed by W.-X. Ma, can produce nonintegrable systems as well. This phenomenon is demonstrated and explained by the example of the enlarged spectral problem of the Korteweg–de Vries equation.

## 1 Introduction

Recently, Ma [1] proposed a method of obtaining new integrable systems of coupled equations through enlarging spectral problems of known integrable equations, and successfully applied it to the AKNS hierarchy. More recently, Ma [2] applied this method to the hierarchy of vector AKNS equations, also successfully. The new hierarchies were proven to be integrable in the sense of possessing recursion operators and infinite sets of commuting symmetries.

The method of Ma consists in using the enlarged first-order Lax pair

$$\Psi_x = X\Psi, \quad \Psi_t = T\Psi \tag{1}$$

with the square matrices  $X$  and  $T$  of the block form

$$X = \begin{pmatrix} U & A \\ 0 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} V & B \\ 0 & 0 \end{pmatrix}, \tag{2}$$

where the square submatrices  $U$  and  $V$  correspond to the Lax pair  $\Phi_x = U\Phi$  and  $\Phi_t = V\Phi$  of a known integrable equation, and the submatrices  $A$  and  $B$  contain additional dependent variables. The compatibility condition

$$D_t X = D_x T - [X, T] \quad (3)$$

of the linear equations (1) is equivalent to

$$D_t U = D_x V - [U, V], \quad (4)$$

$$D_t A = D_x B - U B + V A, \quad (5)$$

where  $D_t$  and  $D_x$  stand for the total derivatives, and the square brackets denote the commutator. Therefore the condition (3) is a zero-curvature representation (ZCR) of a coupled system which consists of the initial integrable equation determined by (4) and an additional equation determined by (5).

In the present paper, we show that — and explain why — the method of Ma can produce nonintegrable coupled equations. We enlarge the spectral problem of the Korteweg–de Vries (KdV) equation and study the evolutionary systems possessing the ZCR (3) with the matrix  $X$  of the form

$$X = \begin{pmatrix} 0 & 1 & p \\ u + \lambda & 0 & q \\ 0 & 0 & 0 \end{pmatrix}, \quad (6)$$

where  $u$ ,  $p$  and  $q$  are functions of  $x$  and  $t$ , and  $\lambda$  is the spectral parameter. Throughout the paper, we set  $T_{31} = T_{32} = T_{33} = 0$ , thus considering matrices  $T$  of the block form (2) only, in compliance with the method of Ma.

In section 2, we choose the matrix  $T$  to be of a definite special form, such that the ZCR (3) determines a one-parameter class of coupled KdV-type equations, and show by means of the Painlevé analysis that at least ten systems of that class cannot be expected to be integrable by the inverse scattering transform technique. In section 3, we obtain the complete class of systems of local evolution equations possessing Lax pairs (1) with  $X$  given by (6) and traceless matrices  $T$  of the block form (2), and observe that most of those systems cannot be integrable in any reasonable sense. In section 4, we explain this strangeness of the studied spectral problem by the existence of a gauge transformation which merges two of the three dependent variables of the matrix  $X$  (6) into one new dependent variable. Section 5 contains concluding remarks.

## 2 Singularity analysis

Let us choose the matrix  $T$  to be of the following special form:

$$T = \begin{pmatrix} u_x & -2u + 4\lambda & T_{13} \\ u_{xx} - 2u^2 + 2\lambda u + 4\lambda^2 & -u_x & T_{23} \\ 0 & 0 & 0 \end{pmatrix} \quad (7)$$

with

$$\begin{aligned} T_{13} &= 4p_{xx} + 4q_x - 2up + 4\lambda p, \\ T_{23} &= 4q_{xx} - pu_x + kup_x - 2uq + 4\lambda p_x + 4\lambda q, \end{aligned} \quad (8)$$

where  $k$  is a parameter. Then the ZCR (3) with the matrices  $X$  (6) and  $T$  (7)–(8) determines the following one-parameter class of triangular systems of coupled KdV-type equations:

$$\begin{aligned} u_t &= u_{xxx} - 6uu_x, \\ p_t &= 4p_{xxx} - (k+2)up_x, \\ q_t &= 4q_{xxx} - 6uq_x - 3qu_x + (k-4)up_{xx} + (k-1)u_xp_x. \end{aligned} \quad (9)$$

In order to show that the method of Ma can produce nonintegrable coupled equations, it is sufficient to find any value of the parameter  $k$  such that the system (9) fails to pass some reliable test for integrability. We choose the Painlevé test in its version for partial differential equations [3, 4]. More precisely, we use the presence of nondominant logarithmic singularities in solutions of a nonlinear system as an undoubted sign of its nonintegrability by the inverse scattering transform technique [5, 6].

Trying to represent the singular behavior of solutions of the system (9) by the expansions

$$u = \sum_{i=0}^{\infty} u_i(t)\phi^{i+\alpha}, \quad p = \sum_{i=0}^{\infty} p_i(t)\phi^{i+\beta}, \quad q = \sum_{i=0}^{\infty} q_i(t)\phi^{i+\gamma} \quad (10)$$

with  $\partial_x\phi(x, t) = 1$ , we find that  $\alpha = -2$ ,  $\beta = -b$  and  $\gamma = -b - 1$ , and that the positions of resonances are  $i = -1, 0, 4, 6, b, b, b + 2, b + 4, 2b + 3$ , where  $b$  is related to the parameter  $k$  by

$$k = 2b^2 + 6b + 2. \quad (11)$$

Non-integer exponents of the dominant behavior of solutions and non-integer positions of resonances not necessarily imply nonintegrability: for example, the integrable Harry Dym equation possesses such analytic properties. Since our aim is to find any undoubtedly nonintegrable case of (9), we continue the singularity analysis for integer values of  $b$  only, in order to discover the incompatibility of recursion relations at resonances, which is the strongest indication of nonintegrability.

Substituting the expansions (10) to the system (9), we get the following recursion relations for their coefficients  $u_i, p_i, q_i$ :

$$\begin{aligned}
& (n-2)(n-3)(n-4)u_n - 3(n-4) \sum_{i=0}^n u_i u_{n-i} \\
& - (n-4)\phi_t u_{n-2} - u_{n-3,t} = 0, \\
& 4(n-b)(n-b-1)(n-b-2)p_n \\
& - (k+2) \sum_{i=0}^n (n-b-i)u_i p_{n-i} \\
& - (n-b-2)\phi_t p_{n-2} - p_{n-3,t} = 0, \\
& 4(n-b-1)(n-b-2)(n-b-3)q_n \\
& - \sum_{i=0}^n (6n-6b-3i-12)u_i q_{n-i} \\
& + \sum_{i=0}^n (n-b-i)((k-4)(n-b)-3k+3i+6)u_i p_{n-i} \\
& - (n-b-3)\phi_t q_{n-2} - q_{n-3,t} = 0,
\end{aligned} \tag{12}$$

where  $n = 0, 1, 2, \dots$ , and  $u_i = p_i = q_i = 0$  for  $i < 0$ . Let us check the compatibility of these recursion relations for  $b = 1, 2, 3, \dots$ . In the case of  $b = 1$ , when  $k = 10$  due to (11), the nontrivial compatibility condition  $p_0 \phi_{tt} = 0$  appears for (12) at the double resonance  $i = 5$ . This means that we have to modify the expansions (10) by introducing additional logarithmic terms, starting from the position  $i = 5$  in this case. Moreover, in at least nine further cases, with  $b = 2, 3, \dots, 10$ , the recursion relations (12) turn out to be incompatible as well: nontrivial compatibility conditions appear there in the position  $i = b$ . For example, we get the condition  $p_0 \phi_t = 0$

at the double resonance  $i = 2$  when  $b = 2$ , the condition  $p_{0,t} = 0$  at the double resonance  $i = 3$  when  $b = 3$ , the condition  $160u_4p_0 - 3p_0\phi_t^2 = 0$  at the triple resonance  $i = 4$  when  $b = 4$ , the condition  $1675p_0\phi_{tt} - 52p_{0,t}\phi_t = 0$  at the double resonance  $i = 5$  when  $b = 5$ , the condition  $166053888u_6p_0 + 44876160u_4p_0\phi_t - 446631p_0\phi_t^3 - 572p_{0,tt} = 0$  at the triple resonance  $i = 6$  when  $b = 6$ , and so on.

Consequently, at least ten systems of the class (9) — namely, those with  $k = 10, 22, 38, 58, 82, 110, 142, 178, 218, 262$  — cannot be expected to be integrable by the inverse scattering transform technique, due to the presence of nondominant logarithmic singularities in their solutions. We have found that the Lax pair (1) with the matrices  $X$  (6) and  $T$  (7)–(8) behaves ill: it represents nonintegrable systems, like the so-called weak Lax pairs do, which contain no essential parameter (see, e.g., § 5.1 in [7] about the nonintegrable Dodd–Fordy equation). On the other hand, this Lax pair (1) with (6)–(8) should be called strong, because it contains the essential parameter  $\lambda$ . In the next two sections, we study how a strong Lax pair can be weak.

### 3 Continual class

Let us find the complete class of systems of local evolution equations

$$\begin{aligned} u_t &= f(x, t, u, p, q, \dots, u_{x\dots x}, p_{x\dots x}, q_{x\dots x}), \\ p_t &= g(x, t, u, p, q, \dots, u_{x\dots x}, p_{x\dots x}, q_{x\dots x}), \\ q_t &= h(x, t, u, p, q, \dots, u_{x\dots x}, p_{x\dots x}, q_{x\dots x}) \end{aligned} \tag{13}$$

which possess the ZCR (3) with the predetermined matrix  $X$  (6) and any traceless  $(2+1) \times (2+1)$ -dimensional matrix  $T$  of the block form (2). It is convenient to solve this problem by the cyclic basis method [8, 9].

The characteristic form of the ZCR (3) is

$$fC_u + gC_p + hC_q = \nabla T, \tag{14}$$

where  $C_u = \partial X / \partial u$ ,  $C_p = \partial X / \partial p$  and  $C_q = \partial X / \partial q$ , and the operator  $\nabla$  is defined by  $\nabla Y = D_x Y - [X, Y]$  for any  $3 \times 3$  matrix  $Y$ . The linearly independent matrices  $C_u$ ,  $C_p$ ,  $C_q$ ,  $\nabla C_u$  and  $\nabla^2 C_u$  constitute the cyclic basis with the closure equations

$$\begin{aligned} \nabla C_p &= -(u + \lambda)C_q, & \nabla C_q &= -C_p, \\ \nabla^3 C_u &= 2u_x C_u - 3(p_x + q)C_p + (p_{xx} + q_x)C_q + 4(u + \lambda)\nabla C_u. \end{aligned} \tag{15}$$

This five-dimensional cyclic basis is sufficient for decomposing any  $3 \times 3$  matrix  $T$  with  $T_{31} = T_{32} = T_{33} = T_{11} + T_{22} = 0$  over it, as follows:

$$T = a_1 C_u + a_2 C_p + a_3 C_q + a_4 \nabla C_u + a_5 \nabla^2 C_u. \quad (16)$$

Then, from (14), (16) and (15), we find that

$$a_1 = D_x^2 a_5 - 4(u + \lambda) a_5, \quad a_4 = -D_x a_5, \quad (17)$$

and that

$$f = D_x^3 a_5 - 4(u + \lambda) D_x a_5 - 2u_x a_5, \quad (18)$$

$$g = D_x a_2 - a_3 - 3(p_x + q) a_5, \quad (19)$$

$$h = -(u + \lambda) a_2 + D_x a_3 + (p_{xx} + q_x) a_5. \quad (20)$$

The relations (18)–(20) determine all the sought systems (13) in terms of all functions  $a_2, a_3, a_5$  of  $\lambda, x, t, u, p, q, \dots, u_{x \dots x}, p_{x \dots x}, q_{x \dots x}$  such that the conditions  $\partial f / \partial \lambda = \partial g / \partial \lambda = \partial h / \partial \lambda = 0$  are satisfied.

It is helpful to rewrite the relations (18)–(20) in the form

$$F = (K - \lambda L) A, \quad (21)$$

where

$$F = \begin{pmatrix} f \\ g \\ h \end{pmatrix}, \quad A = \begin{pmatrix} a_2 \\ a_3 \\ a_5 \end{pmatrix}, \quad (22)$$

$$K = \begin{pmatrix} 0 & 0 & D_x^3 - 4u D_x - 2u_x \\ D_x & -1 & -3p_x - 3q \\ -u & D_x & p_{xx} + q_x \end{pmatrix}, \quad (23)$$

$$L = \begin{pmatrix} 0 & 0 & 4D_x \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \quad (24)$$

If the cyclic basis method, being applied to a certain spectral problem, leads to a relation of the form (21) and the operator  $L^{-1}$  exists there, then one can immediately conclude that the class of local evolutionary systems associated with that spectral problem is a discrete hierarchy with the recursion operator  $R = KL^{-1}$  [9, 10]. Indeed, by using the expansion  $A = A_0 + \lambda A_1 + \lambda^2 A_2 + \dots$

(if  $A$  is not analytic at  $\lambda = 0$ , a shift of  $\lambda$  is required, with no effect on  $L$ ) one finds from (21) and  $\partial F/\partial\lambda = 0$  that  $F = KA_0$  and  $KA_{i+1} = LA_i$  ( $i = 0, 1, 2, \dots$ ), which show that  $F = KL^{-1}F'$  is the right-hand side of a represented system if and only if  $F'$  is. For example, one can easily prove that the operator  $L^{-1}$  exists in the case of the enlarged AKNS spectral problem studied in [1], and that the operator  $R = KL^{-1}$  is in that case exactly the recursion operator obtained in [1]. However, the present case of the enlarged KdV equation's spectral problem is manifestly different: we see from (24) that the operator  $L^{-1}$  does not exist.

In order to overcome this observed degeneracy of the relations (18)–(20), we use (19) as a definition of  $a_3$ ,

$$a_3 = D_x a_2 - 3(p_x + q)a_5 - g, \quad (25)$$

and then obtain from (18) and (20) the following:

$$\begin{pmatrix} f \\ s \end{pmatrix} = \begin{pmatrix} M - \lambda N \end{pmatrix} \begin{pmatrix} a_2 \\ a_5 \end{pmatrix}, \quad (26)$$

where

$$M = \begin{pmatrix} 0 & D_x^3 - 4uD_x - 2u_x \\ D_x^2 - u & -3vD_x - 2v_x \end{pmatrix}, \quad (27)$$

$$N = \begin{pmatrix} 0 & 4D_x \\ 1 & 0 \end{pmatrix}, \quad (28)$$

$$s = D_x g + h, \quad v = p_x + q. \quad (29)$$

Since the operator  $N^{-1}$  exists for  $N$  (28), we can immediately conclude from (26) that the pairs of functions  $f$  and  $s$  satisfying the conditions  $\partial f/\partial\lambda = \partial s/\partial\lambda = 0$  constitute a discrete hierarchy generated by the recursion operator  $R = 4MN^{-1}$  (the factor 4 is taken for simplicity):

$$R = \begin{pmatrix} D_x^2 - 4u - 2u_x D_x^{-1} & 0 \\ -3v - 2v_x D_x^{-1} & 4D_x^2 - 4u \end{pmatrix}. \quad (30)$$

Note, however, that the function  $g$  remains arbitrary because no restriction appeared on it.

The operator  $R$  (30) is the Gürses–Karasu recursion operator of the new integrable coupled KdV system

$$u_t = u_{xxx} - 6uu_x, \quad v_t = 4v_{xxx} - 6uv_x - 3vu_x \quad (31)$$

discovered in [11]. It was shown in [12] that the system (31) passes the Painlevé test well and possesses a  $(3 \times 3)$ -dimensional Lax pair (equivalent to a special case of the enlarged KdV equation's Lax pair, actually).

Thus, we have found the following continual class of evolutionary systems (13) possessing the enlarged Lax pair (1)–(2) with  $X$  determined by (6):

$$u_t = f, \quad p_t = g, \quad q_t = s - D_x g, \quad (32)$$

where

$$\begin{pmatrix} f \\ s \end{pmatrix} = R^n \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad n = 0, 1, 2, \dots, \quad (33)$$

$R$  is given by (30) with  $v = p_x + q$ , the quantity  $D_x^{-1}0$  is understood as any function of  $t$ , and  $g$  is an arbitrary function of  $x, t, u, p, q$  and  $x$ -derivatives of  $u, p, q$  of any maximal order not related to  $n$ . Though the matrix  $X$  (6) contains the essential parameter  $\lambda$  which cannot be removed by a gauge transformation of the ZCR, as one can see from the gauge-invariant coefficients of the closure equations (15), the obtained class of systems (32) is a continual class containing the arbitrary function  $g$ , not a discrete hierarchy that could be expected in a strong Lax pair case. We have seen in section 2 that some systems of the class (32), due to their analytic properties, cannot be expected to be integrable through the inverse scattering transform. Moreover, we believe that most of systems (32) are nonintegrable in any reasonable sense. Indeed, the evident reduction  $u = 0, q = -p_x$ , made in (32), leads to the class of all scalar evolution equations  $p_t = \tilde{g}(x, t, p, p_x, \dots, p_{x \dots x})$ , but there is no sense to speak about integrability of a generic evolution equation.

## 4 Gauge transformation

The observed strangeness of the studied Lax pair can be explained by the existence of a gauge transformation

$$\begin{aligned} \Psi' &= G\Psi, \quad \det G \neq 0, \\ X' &= GXG^{-1} + (D_x G)G^{-1}, \\ T' &= GTG^{-1} + (D_t G)G^{-1} \end{aligned} \quad (34)$$

which merges two of the three dependent variables of the matrix  $X$  (6) into one new dependent variable. Indeed, the transformation (34) with

$$G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & p \\ 0 & 0 & 1 \end{pmatrix} \quad (35)$$

changes the matrix  $X$  (6) into

$$X' = \begin{pmatrix} 0 & 1 & 0 \\ u + \lambda & 0 & v \\ 0 & 0 & 0 \end{pmatrix}, \quad (36)$$

where  $v = p_x + q$ .

Let us find what the matrix  $T$  of the studied Lax pair is changed into by this gauge transformation. It follows from (16), (17) and (25) that the matrix  $T$  corresponding to the class (32) has the form

$$T = \begin{pmatrix} D_x a_5 & -2a_5 & a_2 - 2pa_5 \\ D_x^2 a_5 - 2(u + \lambda)a_5 & -D_x a_5 & T_{23} \\ 0 & 0 & 0 \end{pmatrix}, \quad (37)$$

where

$$T_{23} = D_x a_2 - p D_x a_5 - 2(p_x + q)a_5 - g. \quad (38)$$

Applying the gauge transformation (34) with  $G$  (35) to (37)–(38), we get

$$T' = \begin{pmatrix} D_x a_5 & -2a_5 & a_2 \\ D_x^2 a_5 - 2(u + \lambda)a_5 & -D_x a_5 & D_x a_2 - 2va_5 + p_t - g \\ 0 & 0 & 0 \end{pmatrix}. \quad (39)$$

It is easy to see from (26)–(28), (30) and (33) that  $a_2$  and  $a_5$  are functions of  $\lambda, u, v, u_x, v_x, \dots, u_{x \dots x}, v_{x \dots x}$  only. Therefore the transformed matrix  $T'$  (39) contains  $p$  and  $q$  in the form of the new dependent variable  $v = p_x + q$  everywhere, except for the expression  $p_t - g$  in  $T'_{23}$ , but  $p_t - g = 0$  on all solutions of any represented system (32).

Consequently, up to the gauge equivalence and the equivalence on solutions [8], the studied ‘strong-and-weak’ enlarged Lax pair effectively contains only one additional dependent variable but not two.

## 5 Conclusion

In the present paper, we studied the method of W.-X. Ma of obtaining new integrable systems of coupled equations through enlarging spectral problems of known integrable equations. We enlarged the spectral problem of the KdV equation and found that the method of Ma can produce nonintegrable systems as well. We explained this strangeness of the enlarged KdV equation's spectral problem by the existence of a gauge transformation which merges two of the three dependent variables of the enlarged spectral problem into one new dependent variable. Let us remind that for the first time a phenomenon of this kind, that nonintegrable systems can possess a ZCR containing an essential parameter, was observed and explained through gauge transformations in [8].

## References

- [1] W.-X. Ma. Enlarging spectral problems to construct integrable couplings of soliton equations. *Phys. Lett. A*, vol. 316, no. 1–2, pp. 72–76 (2003).
- [2] W.-X. Ma. Integrable couplings of vector AKNS soliton equations. *J. Math. Phys.*, vol. 46, no. 3, art. 033507, 19 pp. (2005).
- [3] J. Weiss, M. Tabor, G. Carnevale. The Painlevé property for partial differential equations. *J. Math. Phys.*, vol. 24, no. 3, pp. 522–526 (1983).
- [4] M. Jimbo, M. D. Kruskal, T. Miwa. Painlevé test for the self-dual Yang–Mills equation. *Phys. Lett. A*, vol. 92, no. 2, pp. 59–60 (1982).
- [5] M. Tabor. *Chaos and Integrability in Nonlinear Dynamics. An Introduction*. Wiley, New York, 1989.
- [6] M. J. Ablowitz, P. A. Clarkson. *Solitons, Nonlinear Evolution Equations and Inverse Scattering*. Cambridge University Press, Cambridge, 1991.
- [7] P. A. Clarkson, C. M. Cosgrove. Painlevé analysis of the non-linear Schrödinger family of equations. *J. Phys. A: Math. Gen.*, vol. 20, no. 8, pp. 2003–2024 (1987).
- [8] S. Yu. Sakovich. On zero-curvature representations of evolution equations. *J. Phys. A: Math. Gen.*, vol. 28, no. 10, pp. 2861–2869 (1995).

- [9] S. Yu. Sakovich. Cyclic bases of zero-curvature representations: five illustrations to one concept. *Acta Appl. Math.*, vol. 83, no. 1–2, pp. 69–83 (2004); arXiv: nlin.SI/0212019.
- [10] S. Yu. Sakovich. On integrability of one third-order nonlinear evolution equation. *Phys. Lett. A*, vol. 314, no. 3, pp. 232–238 (2003); arXiv: nlin.SI/0303040.
- [11] M. Gürses, A. Karasu. Integrable coupled KdV systems. *J. Math. Phys.*, vol. 39, no. 4, pp. 2103–2111 (1998); arXiv: solv-int/9711015.
- [12] S. Yu. Sakovich. Coupled KdV equations of Hirota–Satsuma type. *J. Nonlinear Math. Phys.*, vol. 6, no. 3, pp. 255–262 (1999); arXiv: solv-int/9901005.